Flirting with Monsky

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Abstract

Monsky's theorem of 1970 says that we cannot dissect a unit square into an odd number of triangles of equal area. A related question, dissecting a rectangle into three triangles of equal area, illustrates the importance of the curvature of space. We prove that the dissection can be done in hyperbolic space, the non-Euclidean geometry of negative curvature.

Introduction

Any excuse to bring up Monsky's theorem is worth the time because the theorem is one clear sentence with a famously clever proof. Monsky's theorem says that we cannot tile a unit square with an odd number of triangles of equal area. The proof assumes a tiling of n triangles with equal area, assigns colors from a set of three colors to the vertices in a reasonable way and shows the existence of at least one triangle with all three colors. The knock-out punch happens when the three-color triangle cannot have area $\frac{1}{n}$ while n is odd. [1] The Euclidean area formula of a triangle is a crucial part of the proof of Monsky's theorem. As a preview to our main ideas, we will follow Monsky's proof for three triangles.

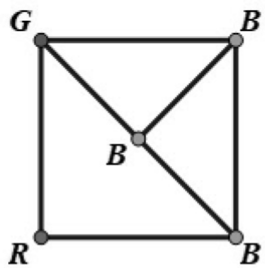

Figure 1. Dissect a square.

Monsky's famous proof applied to three triangles conveys only some of the cleverness. Since we want equal area, each triangle has to have area $\frac{1}{3}$. We note right away that the actual (x, y) coordinates of all the points have to be rational numbers, no square roots or cube roots allowed because radicals are sticky: once an area calculation involves radicals, the radicals usually stick around and we will not get the area to be $\frac{1}{3}$. As a case in point, there is an area formula for a triangle with vertices at (x_1, y_1) and (x_2, y_2) and (0,0): $\frac{1}{2}\left|x_1y_2 - x_2y_1\right|$. If any coordinates had radicals, the area would have radicals.

Monsky wrote down a rule to use which assigns colors to points according to their coordinates. The rule was followed in Figure 1. The three colors are **R**ed, **G**reen and **B**lue (marked R, G, B.) His rule fit Sperner's Lemma, which claimed that there would always be an odd number of triangles with all three colors assigned to vertices. Monsky also noted that the square could be manipulated so that one of these tri-color triangles has a vertex at the origin, as in our Figure 1. Our area calculation for that triangle is an easy $\frac{1}{2}$. Monsky had to deal with the general case, not knowing the exact coordinates of the tri-color triangle. He was able to show that its area had to be $\frac{1}{m}$ with m even (in our case, $m = 2$.) The general case requiring area $\frac{1}{n}$, n odd was thus proved unobtainable. The proof is famous for his dealing with all possible cases in one shot, with no drawing to stand for all cases. Even for a civilized number of triangles like 15, there would be approximately a zillion ways to draw the little triangles; in other words, a case-by-case proof would have been crazy to even consider.

The reader may correctly regard the above argument as cheesy, since our tri-color triangle never had a chance to have area $\frac{1}{3}$. That is actually the point of Monsky's theorem: any suitable drawing would be impossible!

Getting around Monsky

Since area calculations in non-Euclidean geometries do not use the Euclidean area formula, we cannot help but wonder about non-Euclidean cases. While we do not challenge Monsky itself, we pose a Monsky-like question: can we tile a rectangle with three triangles of equal area?

In this article, we try this question in three geometries: Euclidean, hyperbolic, and elliptic. We will find only hyperbolic geometry allows such a dissection.

We define a rectangle as an equiangular quadrilateral. That way, we can think about rectangles in all three geometries. In all three geometries, our figure has three interior faces, all triangles. The Euler characteristic formula forces a relationship between the number of vertices and edges.

$$V - E + F = V - E + 3 = 1 \qquad E - V = 2.$$

A suitable arrangement of three triangles filling a rectangle has to be one of these two in Figure 2. The vertex which is common to all three triangles can be on the boundary or inside the rectangle. Adding another edge can only increase the number of vertices by 1. We cannot add an edge so that another face is created. All we can do is append a little antenna to an existing vertex and stick a vertex on the end of that new edge, as in Figure 2. But that does not change our triangulation. Now we have to look for sizes which give us three triangles of equal area.

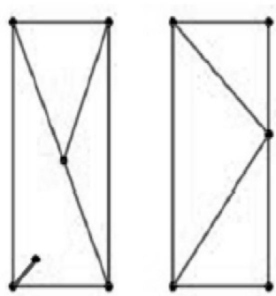

Figure 2.

The Euclidean case is impossible regardless of the rectangle's dimensions. No matter how the options are scaled, the area of the largest of the three triangles will equal the area of the other two combined.

Elliptic geometry is the non-Euclidean geometry of positive curvature, like a sphere. The sum of the angles of an elliptic triangle is greater than π radians. The area of an elliptic triangle is the sum of its angles minus π. Although we should not judge elliptic areas by appearance alone, those areas are like areas on a globe. Figure 3 shows our best candidate for the elliptic version where we try to get triangle *CMB* the same area as triangle *MBA*: get the triangles tall and thin. The segments $\overline{MC}$ and $\overline{MB}$ bend away from the center, making that middle triangle bigger than the sum of the areas of the other two triangles, a situation actually worse than the Euclidean, because the area of the largest triangle will be greater than the sum of the other two triangles.

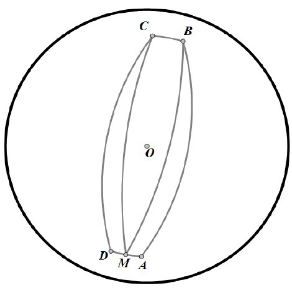

Figure 3.

The hyperbolic example in Figure 4, however, gives us three triangles of equal area which tile a hyperbolic rectangle. Hyperbolic geometry is the non-Euclidean geometry of negative curvature. For our example, we have used the Poincaré model of hyperbolic geometry. Hyperbolic space is the set of points inside the large circle. The four hyperbolic lines are arcs of Euclidean circles orthogonal to the boundary circle. The two hyperbolic segments inside the rectangle are also on arcs of circles orthogonal to the boundary; we didn't draw their entire arcs. Angles BMA and CMD are $\frac{\pi}{4}$. The vertex angles are all $\frac{\pi}{4}$ and M is the midpoint of $\overline{AD}$. (The two disk models of our non-Euclidean geometries are presented in [2]).

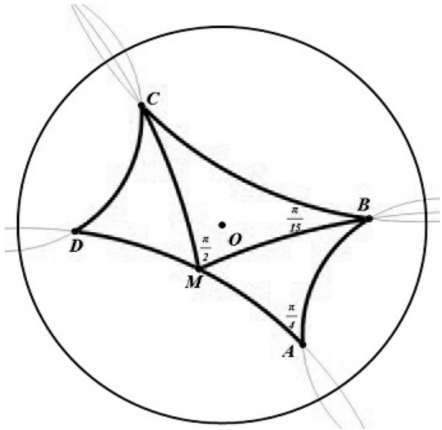

Figure 4. Hyperbolic solution.

The hyperbolic area of a triangle is π minus the sum of the angles. So, like the elliptic case, we want the sum of the angles of each triangle to be the same. Unlike the previous two cases, appearances are not much use in judging hyperbolic area because a little area near the boundary counts as a lot more area than the same-looking area near the center. There is a hyperbolic trigonometric formula for triangle *ABC* with side lengths *a, b, c* opposite angles *A, B, C*.

$$\sin A \sin B \cosh c = \cos C + \cos A \cos B \tag{1}$$

First, the angle sums of the outer two triangles are $2(\frac{\pi}{4}) + \frac{\pi}{6}$ and the middle triangle has sum $\frac{\pi}{2} + 2(\frac{\pi}{12})$, which is $\frac{2\pi}{3}$ for each triangle. Now, we must verify that the length $M\,B$ is the same for both triangles which have side $\overline{MB}$. For the triangle *ABM*, this measurement in (1) gives us

$$\sin\tfrac{\pi}{4}\sin\tfrac{\pi}{6}\cosh MB = \cos\tfrac{\pi}{4} + \cos\tfrac{\pi}{4}\cos\tfrac{\pi}{6}\,,$$

which implies cosh $MB = \dfrac{2\sqrt{2}+\sqrt{6}}{\sqrt{2}} = 2+\sqrt{3}$. For the triangle *BMC*, (1) becomes

$$\sin\tfrac{\pi}{12}\sin\tfrac{\pi}{2}\cosh MB = \cos\tfrac{\pi}{12} + \cos\tfrac{\pi}{12}\cos\tfrac{\pi}{2} = \cos\tfrac{\pi}{12}.$$

We obtain the sine and cosine of $\dfrac{\pi}{12}$ from trigonometric difference formulas:

$$\cos\tfrac{\pi}{12} = \cos(\tfrac{\pi}{4} - \tfrac{\pi}{6}) = \cos\tfrac{\pi}{4}\cos\tfrac{\pi}{6} + \sin\tfrac{\pi}{4}\sin\tfrac{\pi}{6} = \frac{\sqrt{6}+\sqrt{2}}{4} \text{ and}$$

$$\sin\tfrac{\pi}{12} = \sin(\tfrac{\pi}{4} - \tfrac{\pi}{6}) = \sin\tfrac{\pi}{4}\cos\tfrac{\pi}{6} - \sin\tfrac{\pi}{6}\cos\tfrac{\pi}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}.$$

These lead to $\cosh MB = \dfrac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}} = 2+\sqrt{3}$, as required.

The angles also determine the lengths of the sides of the hyperbolic rectangle *ABCD*. We now verify the hyperbolic lengths *AD* and *BC* are equal by calculation. For triangle *BCM,* (1) becomes

$$\begin{aligned} \sin^2 \tfrac{\pi}{12} \cosh BC &= \cos \tfrac{\pi}{2} + \cos^2 \tfrac{\pi}{12} \\ \frac{8-4\sqrt{3}}{16} \cosh BC &= \frac{8+4\sqrt{3}}{16} \\ \cosh BC &= \frac{2+\sqrt{3}}{2-\sqrt{3}} = 7+4\sqrt{3}. \end{aligned}$$

For triangle *ABM,* (1) becomes

$$\begin{aligned} \sin^2 \tfrac{\pi}{4} \cosh MA &= \cos \tfrac{\pi}{6} + \cos^2 \tfrac{\pi}{4} \\ \tfrac{1}{2} \cosh MA &= \tfrac{\sqrt{3}}{2} + \tfrac{1}{2} \\ \cosh MA &= \sqrt{3} + 1. \end{aligned}$$

The double-angle formula for $\cosh(2MA) = \cosh AD$ gives us

$$\cosh(2MA) = 2\cosh^2 MA - 1 = 2(\sqrt{3}+1)^2 - 1 = 8 + 4\sqrt{3} - 1 = 7 + 4\sqrt{3}.$$

The sides $\overline{CD}$ and $\overline{AB}$ have to be the same size because triangle *ABM* is congruent to triangle *DCM.* These three triangles really do fit together to form a hyperbolic rectangle.

The interesting idea is that the two segments $\overline{MB}$ and $\overline{MC}$ bulge in the direction which decreases the area of the middle triangle, which is our only hope in cracking the case. Yet, the arcs we refer to as lines in the non-Euclidean models are actually straight in hyperbolic geometry. The hyperbolic segments do not actually bulge in hyperbolic space! Our Euclidean eyes allow us to see a useful attribute of hyperbolic space: we need negative curvature to dissect a rectangle into three triangles of equal area.

Acknowledgement

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